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Subduction of dominant representations for combinatorial enumeration

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Summary. A new method for giving cycle indices is presented for combinatorial enumeration. Thus, cyclic groups are characterized by markaracter tables, the elements of which are determined by the orders of their subgroups. A set of such cyclic groups (defined as dominant subgroups) is used to characterize a group **G** of finite order, where the markaracter table for the group **G** is constructed with respect to dominant representations (DRs), which are defined as coset representations corresponding to the dominant subgroups. By starting from the markaracter table, we propose an essential set of subdominant markaracter tables and a magnification set for the group **G**; the latter concept clarifies the relationship between each subdominant markaracter table and the markaracter table of a dominant subgroup. The subduction of DRs is obtained by the markaracter table to produce a dominant subduction table and a dominant USCI (unit-subduced cycle index) table. The latter is used to evaluate a cycle index to be applied to combinatorial enumeration. The cycle index is shown to be equivalent to the couterpart of our previous approach concerning both cyclic and non-cyclic subgroups. The latter, in turn, has been proved to be equivaltent to the cycle index obtained by the Redfield-Pólya theorem.

Key words: Dominant representation – Subduction – Enumeration – Markaracter

1. Introduction

The Redfield-Pólya theorem has been a standard method for combinatorial enumeration in chemistry as well as in mathematics [1, 2]. This method uses a cycle index as a key concept to evaluate isomer numbers. The cycle index is a polynomial which contains a term determined for each conjugacy class. Other approaches that create a cycle index equivalent to the Redfield-Pólya theorem have been proposed; *e.g.* a doube-coset method by Ruch *et al.* [3] and a method using mark tables [4]. We have reported the USCI approach and an alternative formulation of the cycle index [5]. Our cycle index is determined on the basis of unit subduced cycle indices (USCI) which correspond to conjugate cyclic subgroups. Although we also take account of terms concerning non-cyclic subgroups, these terms vanish in the process of calculating a cycle index. If we focus on the relationship between conjugacy classes and conjugate cyclic subgroups, our previous formulation can be simplified so that we use conjugate cyclic subgroups only. In the preceding article, we have discussed dominant representations (DRs) based on such cyclic subgroups. We here deal with several properties of markaracter tables and with subduction of DRs. The final target of this article is to propose a novel formulation of combinatorial enumeration as a simplified USCI approach.

2. Essential set of subdominant markaracter tables

2.1. Markaracter tables for cyclic groups

In the preceding paper, a group $\mathbf{G}_j$ of finite order is characterized by a markaracter table. When the group $\mathbf{G}_j$ is a cyclic group, the markaracter table of $\mathbf{G}_j$ is equal to its mark table. Moreover, each subgroup of $\mathbf{G}_j$ is a cyclic subgroup whose order is equal to a divisor of $|\mathbf{G}_j|$.

Let $\mathbf{G}_j$ be a cyclic group. The group $\mathbf{G}_j$ has a non-redundant set of cyclic subgroups, *i.e.*,

$$SCSG_{\mathbf{G}_j} = \{\mathbf{G}_1^{(j)}, \mathbf{G}_2^{(j)}, \ldots, \mathbf{G}_r^{(j)}\}, \tag{1}$$

where the group $\mathbf{G}_1^{(j)}$ is an identity group and $\mathbf{G}_r^{(j)}$ is identical with $\mathbf{G}_j$ istself.

Since the group $\mathbf{G}_j$ is cyclic, the normalizers of its subgroups are identical with each other, *i.e.*,

$$\mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_1^{(j)}) = \mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_2^{(j)}) = \cdots = \mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_r^{(j)}) = \mathbf{G}_j. \tag{2}$$

When we focus our attention on a cyclic group, Theorems 2 to 5 in the preceding article are summarized into the following simple lemma,

Lemma 1. *The dominant markaracter (*i.e. *the k-th row of the markaracter table) of the cyclic group G_j is represented by*

$$\begin{aligned} \mathbf{G}_j(/\mathbf{G}_k^{(j)}) &= (\lambda_{k1}^{(j)}, \lambda_{k2}^{(j)}, \ldots, \lambda_{k\ell}^{(j)}, \ldots, \lambda_{kk}^{(j)}, \ldots, \lambda_{kr}^{(j)}) \\ &= (\lambda_{k1}^{(j)}, \lambda_{k2}^{(j)}, \ldots, \lambda_{k\ell}^{(j)}, \ldots, \lambda_{kk}^{(j)}, 0, \ldots, 0), \end{aligned} \tag{3}$$

where each element is expressed by

$$\lambda_{k\ell}^{(j)} = \begin{cases} = \frac{|\mathbf{G}_j|}{|\mathbf{G}_k^{(j)}|} & \text{for } \mathbf{G}_\ell^{(j)} (\leq \mathbf{G}_k^{(j)}) \quad \text{(4a)} \\ = 0 & \text{for } \mathbf{G}_\ell^{(j)} (\not\leq \mathbf{G}_k^{(j)}) \quad \text{(4b)} \end{cases}$$

for $\ell = 1, 2, \ldots, k$.

Such markaracters construct a markaracter table for the cyclic group $\mathbf{G}_j$. Thus, we have $\widetilde{\mathbf{M}}_{\mathbf{G}_j} = (\lambda_{k\ell}^{(j)})$ as a lower triangular $r \times r$ matrix.

The following markaracter tables for $\mathbf{C}_2$, $\mathbf{C}_s$, $\mathbf{C}_3$, and $\mathbf{S}_4$ exemplify Lemma 1.

$$\widetilde{\mathbf{M}}_{\mathbf{C}_2} = \begin{array}{c} \\ \mathbf{C}_2(/\mathbf{C}_1) \\ \mathbf{C}_2(/\mathbf{C}_2) \end{array} \begin{array}{c} \begin{array}{cc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 \end{array} \\ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \end{array} \tag{5}$$

$$\widetilde{\mathbf{M}}_{\mathbf{C}_s} = \begin{array}{c} \\ \mathbf{C}_s(/\mathbf{C}_1) \\ \mathbf{C}_s(/\mathbf{C}_s) \end{array} \begin{array}{c} \begin{array}{cc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_s \end{array} \\ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \end{array} \tag{6}$$

$$\widetilde{\mathbf{M}}_{\mathbf{C}_3} = \begin{array}{c} \\ \mathbf{C}_3(/\mathbf{C}_1) \\ \mathbf{C}_3(/\mathbf{C}_3) \end{array} \begin{array}{c} \begin{array}{cc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_3 \end{array} \\ \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \end{array} \tag{7}$$

$$\widetilde{\mathbf{M}}_{\mathbf{S}_4} = \begin{array}{c} \\ \mathbf{S}_4(/\mathbf{C}_1) \\ \mathbf{S}_4(/\mathbf{C}_2) \\ \mathbf{S}_4(/\mathbf{S}_4) \end{array} \begin{array}{c} \begin{array}{ccc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{S}_4 \end{array} \\ \begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \tag{8}$$

Since the markaracter table ($\widetilde{\mathbf{M}}_{\mathbf{G}_j} = (\lambda_{k\ell}^{(j)})$) of the cyclic group $\mathbf{G}_j$ is a lower triangular matrix, the corresponding inverse ($\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} = (\overline{\lambda}_{\ell k}^{(j)})$) is also lower triangular. The elements of $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$ are evaluated by the following lemma using the Möbius function.

Lemma 2. *The k-th colomn of the inverse $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$ for the cyclic subgroup $\mathbf{G}_j$ is represented by*

$$\begin{aligned} \overline{\mathbf{G}_j(/\mathbf{G}_k^{(j)})}^T &= (\overline{\lambda}_{1k}^{(j)}, \overline{\lambda}_{2k}^{(j)}, \ldots, \overline{\lambda}_{\ell k}^{(j)}, \ldots, \overline{\lambda}_{kk}^{(j)}, \ldots, \overline{\lambda}_{rk}^{(j)})^T \\ &= (0, \ldots, 0, \overline{\lambda}_{kk}^{(j)}, \overline{\lambda}_{k+1,k}^{(j)}, \ldots, \lambda_{\ell k}^{(j)}, \ldots, \lambda_{rk}^{(j)})^T, \end{aligned} \tag{9}$$

where each element is expressed by

$$\overline{\lambda}_{\ell k}^{(j)} = \begin{cases} = \mu\left(\frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right)\frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} & \textit{for } \mathbf{G}_\ell^{(j)}(\geq \mathbf{G}_k^{(j)}) \quad (10a) \\ = 0 & \textit{for } \mathbf{G}_\ell^{(j)}(\not\geq \mathbf{G}_k^{(j)}) \quad (10b) \end{cases}$$

for $\ell = k, k+1, \ldots, r$.

Proof. Let $\mu(n)$ be the Möbius function of integer n. Since the matrices at issue are lower triangular, the multiplication of the ℓ-th row of $\widetilde{\mathbf{M}}_{\mathbf{G}_j}$ by the k-th colomn of the inverse $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$ gives a non-diagonal element, which vanishes into zero as represented by

$$\begin{aligned} \mathbf{G}_j(/\mathbf{G}_\ell^{(j)})\overline{\mathbf{G}_j(/\mathbf{G}_k^{(j)})}^T &= \frac{|\mathbf{G}_j|}{|\mathbf{G}_\ell^{(j)}|}\sum_{\ell'=k}^{\ell}\mu\left(\frac{|\mathbf{G}_{\ell'}^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right)\frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} \\ &= \frac{|\mathbf{G}_j|}{|\mathbf{G}_\ell^{(j)}|}\frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|}\sum_{\ell'=k}^{\ell}\mu\left(\frac{|\mathbf{G}_{\ell'}^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right) \\ &= \frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_\ell^{(j)}|}\sum_{d|n}\mu(d) = 0, \end{aligned} \tag{11}$$

where $d = \frac{|\mathbf{G}_{\ell'}^{(j)}|}{|\mathbf{G}_k^{(j)}|}$ is a divisor of $n = \frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}$ and the summation designated by $d|n$ runs over all of the divisors (d) of n. Each diagonal element is calculated to be

$$\frac{|\mathbf{G}_j|}{|\mathbf{G}_k^{(j)}|} \times \mu\left(\frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right) \frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} = 1. \tag{12}$$

Equations 11 and 12 mean that $\widetilde{\mathbf{M}}_{\mathbf{G}_j}$ is inverse to $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$. □

The elements involved in $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$ reveal a property of the Möbius function. Thus, the summation of the elements appearing in the k-th column of $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$ is represented by

$$\begin{aligned}\sum_{\ell=k}^{r} \overline{\lambda}_{\ell k}^{(j)} &= \sum_{\ell=k}^{r} \mu\left(\frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right) \frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} = \frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} \sum_{\ell=k}^{r} \mu\left(\frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right) \\ &= \frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} \sum_{d|n} \mu(d) = 0,\end{aligned} \tag{13}$$

where $d = \frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}$ is a divisor of $n = \frac{|\mathbf{G}_r^{(j)}|}{|\mathbf{G}_k^{(j)}|}$ and the summation designated by $d|n$ runs over all of the divisors (d) of n.

On the other hand, the summation of the elements in the ℓ-th row reveals the relationship between the Möbius function and the Euler function. Thus, it is representated by

$$\begin{aligned}\sum_{k=1}^{\ell} \overline{\lambda}_{\ell k}^{(j)} &= \sum_{k=1}^{\ell} \mu\left(\frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right) \frac{|\mathbf{G}_k^{(j)}|}{|\mathbf{G}_j|} = \frac{1}{|\mathbf{G}_j|} \sum_{k=1}^{\ell} \mu\left(\frac{|\mathbf{G}_\ell^{(j)}|}{|\mathbf{G}_k^{(j)}|}\right) |\mathbf{G}_k^{(j)}| \\ &= \frac{\varphi(|\mathbf{G}_\ell^{(j)}|)}{|\mathbf{G}_j|}.\end{aligned} \tag{14}$$

Note that each $|\mathbf{G}_k^{(j)}|$ is a divisor of $|\mathbf{G}_\ell^{(j)}|$ or there appears a zero entry for the row.

Equations 13 and 14 are summarized as a theorem as follows, where the ranges of the summations are rewritten by taking zero entries into consideration.

Theorem 1. *The elements in the k-th comolun of $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} = (\overline{\lambda}_{\ell k}^{(j)})$ for the cyclic group $\mathbf{G}_j$ are summed up to vanish into zero,* i.e.,

$$\sum_{\ell=1}^{r} \overline{\lambda}_{\ell k}^{(j)} = 0. \tag{15}$$

The elements in the ℓ-th row of $\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} = (\overline{\lambda}_{\ell k}^{(j)})$ for the cyclic group $\mathbf{G}_j$ are summed up to give

$$\sum_{k=1}^{r} \overline{\lambda}_{\ell k}^{(j)} = \frac{\varphi(|\mathbf{G}_\ell^{(j)}|)}{|\mathbf{G}_j|}. \tag{16}$$

2.2. *Subdominant markaracter tables for dominant subgroups*

In the treatment described in the preceding article, we select an appropriate set of rows from the markaracter table of a group **G** so as to construct a subdominant markaracter table of its subgroup $\mathbf{G}_j$ which may be cyclic or non-cyclic. In this section, however, we consider a cyclic subgroup $\mathbf{G}_j$ of **G**. This treatment allows us to prove that each row of the markaracter table can be constructed from the date of the corresponding cyclic subgroup.

The group **G** is characterized by a non-redundant set of subdominant representations (SSDR), which are represented by $\mathbf{G} \downarrow \mathbf{G}_j(/\mathbf{G}_k^{(j)})$ for $j = 1, 2, \cdots, s$. Each subdominant representation (SDR) corresponds to a subdominant markaracter, the elements of which are selected from the corresponding markaracter table. This process is permitted on the basis of Theorems 17 and 18 of the preceding paper. Thus, we have a subdominant markaracter, $\mathbf{G} \downarrow \mathbf{G}_j(/\mathbf{G}_k^{(j)})$, which is identical with eq. 63 of the preceding paper. Such subdominant markaracters are collected to form a subdominant markaracter table (SDMT), which is a lower triangular matrix ($\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$), as shown in the preceding paper (eq. 64).

Example 1. The group $\mathbf{T}_d$ has a markaracter table ($\widetilde{\mathrm{M}}_{\mathbf{T}_d}$), which is identical with Table 2 of the preceding paper. For constructing subdominant mark tables, we select the elements of respective cyclic subgroups.

$$\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{T}_d\downarrow\mathbf{C}_2} = \begin{array}{l} \\ \mathbf{T}_d \downarrow \mathbf{C}_2(/\mathbf{C}_1) \\ \mathbf{T}_d \downarrow \mathbf{C}_2(/\mathbf{C}_2) \end{array} \begin{array}{c} \begin{array}{cc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 \end{array} \\ \begin{pmatrix} 24 & 0 \\ 12 & 4 \end{pmatrix} \end{array} \tag{17}$$

$$\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{T}_d\downarrow\mathbf{C}_s} = \begin{array}{l} \\ \mathbf{T}_d \downarrow \mathbf{C}_s(/\mathbf{C}_1) \\ \mathbf{T}_d \downarrow \mathbf{C}_s(/\mathbf{C}_s) \end{array} \begin{array}{c} \begin{array}{cc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_s \end{array} \\ \begin{pmatrix} 24 & 0 \\ 12 & 2 \end{pmatrix} \end{array} \tag{18}$$

$$\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{T}_d\downarrow\mathbf{C}_3} = \begin{array}{l} \\ \mathbf{T}_d \downarrow \mathbf{C}_3(/\mathbf{C}_1) \\ \mathbf{T}_d \downarrow \mathbf{C}_3(/\mathbf{C}_3) \end{array} \begin{array}{c} \begin{array}{cc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_3 \end{array} \\ \begin{pmatrix} 24 & 0 \\ 8 & 2 \end{pmatrix} \end{array} \tag{19}$$

$$\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{T}_d\downarrow\mathbf{S}_4} = \begin{array}{l} \\ \mathbf{T}_d \downarrow \mathbf{S}_4(/\mathbf{C}_1) \\ \mathbf{T}_d \downarrow \mathbf{S}_4(/\mathbf{C}_2) \\ \mathbf{T}_d \downarrow \mathbf{S}_4(/\mathbf{S}_4) \end{array} \begin{array}{c} \begin{array}{ccc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{S}_4 \end{array} \\ \begin{pmatrix} 24 & 0 & 0 \\ 12 & 4 & 0 \\ 6 & 2 & 2 \end{pmatrix} \end{array} \tag{20}$$

These subdominant markaracter tables are compared with the dominant markaracter tables of the respective cyclic subgroups. By inspection, we are able to obtain diagonal matrices to change the latter to the former. The diagonal matrices are called the *magnification set* of the group **G**, since they are uniquely determined if the group **G** is given as proved below.

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 12 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 24 & 0 \\ 12 & 4 \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 24 & 0 \\ 12 & 2 \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 24 & 0 \\ 8 & 2 \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 24 & 0 & 0 \\ 12 & 4 & 0 \\ 6 & 2 & 2 \end{pmatrix} \tag{24}$$

The sencond matrix in the left-hand side of each equation is the magnification of the subgroups at issue. The above discussion can be extended to general cases, since we have evaluated the elements of a markaracter table (Theorems 2 to 5 of the preceding paper). Thus, we have the following lemma.

Lemma 3. *Let $m_{\ell k}^{(j)}$ be the element of the subdominant markaracter table for* $\mathbf{G} \downarrow \mathbf{G}_j$, i.e., $\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$. *Suppose that the element corresponds to the element $\lambda_{\ell k}^{(j)}$ of the markaracter table of the subgroup $\mathbf{G}_j$. When ℓ moves k to r* (i.e. *over the kth column), we have*

$$\frac{m_{\ell k}^{(j)}}{\lambda_{\ell k}^{(j)}} = \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_k^{(j)})|}{|\mathbf{G}_j|} \quad (\ell = k, k+1, \ldots, r) \tag{25}$$

for respective non-zero elements.

Proof. We have the ratio for the diaganonal entry of the k-th column,

$$\frac{m_{kk}^{(j)}}{\lambda_{kk}^{(j)}} = \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_k^{(j)})|}{|\mathbf{G}_k^{(j)}|} \Big/ \frac{|\mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_k^{(j)})|}{|\mathbf{G}_k^{(j)}|} = \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_k^{(j)})|}{|\mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_k^{(j)})|} \tag{26}$$

We then obtain the ratio for the entry at the intersection between the k-th column and the ℓ-the row,

$$\frac{m_{\ell k}^{(j)}}{\lambda_{\ell k}^{(j)}} = \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_\ell^{(j)})|}{|\mathbf{G}_\ell^{(j)}|} \Big/ \frac{|\mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_\ell^{(j)})|}{|\mathbf{G}_\ell^{(j)}|} = \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_\ell^{(j)})|}{|\mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_\ell^{(j)})|}. \tag{27}$$

Since $\mathbf{G}_k$ and $\mathbf{G}_\ell$ are cyclic groups of $\mathbf{G}_j$, we have

$$\mathbf{N}_{\mathbf{G}}(\mathbf{G}_\ell^{(j)}) = \mathbf{N}_{\mathbf{G}}(\mathbf{G}_k^{(j)}), \quad \mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_\ell^{(j)}) = \mathbf{N}_{\mathbf{G}_j}(\mathbf{G}_k^{(j)}) = \mathbf{G}_j \tag{28}$$

These relationships allow us to equalize eqs. 26 and 27.

$$\frac{m_{kk}^{(j)}}{\lambda_{kk}^{(j)}} = \frac{m_{\ell k}^{(j)}}{\lambda_{\ell k}^{(j)}} = \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_k^{(j)})|}{|\mathbf{G}_j|} \quad (\ell = k, k+1, \ldots, r). \tag{29}$$

□

In the light of this lemma, we define a *magnification* as a diagonal matrix represented by

$$\mathrm{L}_{\mathbf{G}_j\uparrow\mathbf{G}} = \begin{pmatrix} \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_1^{(j)})|}{|\mathbf{G}_j|} & & & 0 \\ & \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_2^{(j)})|}{|\mathbf{G}_j|} & & \\ & 0 & \ddots & \\ & & & \frac{|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_r^{(j)})|}{|\mathbf{G}_j|} \end{pmatrix}. \tag{30}$$

Then we have a theorem that allows us to construct a subdominant markaracter table from the markarcter table of the corresponding subgroup.

Theorem 2. *Let* $\widetilde{\mathrm{M}}_{\mathbf{G}_j}$ *be the markaracter table of the cyclic group* $\mathbf{G}_j$. *Suppose that* $\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$ *is the SDMT produced from* $\widetilde{\mathrm{M}}_{\mathbf{G}}$. *When we use the magnification represented by* $\mathrm{L}_{\mathbf{G}_j\uparrow\mathbf{G}}$ *(eq. 30), we have*

$$\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j} = \widetilde{\mathrm{M}}_{\mathbf{G}_j}\mathrm{L}_{\mathbf{G}_j\uparrow\mathbf{G}}, \tag{31}$$

or inversely,

$$\widetilde{\widetilde{\mathrm{M}}}^{-1}_{\mathbf{G}\downarrow\mathbf{G}_j} = \mathrm{L}^{-1}_{\mathbf{G}_j\uparrow\mathbf{G}}\widetilde{\mathrm{M}}^{-1}_{\mathbf{G}_j} \tag{32}$$

The resulting subdominant markaracter table (SDMT) is a part of the markaracter table of the group **G**.

Let **G** be a cyclic group. The group **G** has a non-redundant set of cyclic subgroups, *i.e.*,

$$SCSG_{\mathbf{G}} = \{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_j, \ldots, \mathbf{G}_s\}. \tag{33}$$

Each subgroup $\mathbf{G}_j$ corresponds to eq. 31. This means that the group **G** is characterized by a set of dominant markaracter tables ($\widetilde{\mathrm{M}}_{\mathbf{G}_j}$) and a set of magnifications ($\mathrm{L}_{\mathbf{G}_j\uparrow\mathbf{G}}$). They produce a set of subdominant markaracter tables which are in turn collected to create the markaracter table of **G**.

2.3. Induction and magnification

Let **H** be a non-cyclic subgroup **G**. The group **H** is characterized by a set of dominant markaracter tables ($\widetilde{\mathrm{M}}_{\mathbf{H}_j}$) and a set of magnifications ($\mathrm{L}_{\mathbf{H}_j\uparrow\mathbf{H}}$), where H_j runs over the SCSG for **H**:

$$SCSG_{\mathbf{G}} = \{\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_j, \ldots, \mathbf{H}_u\}, \tag{34}$$

which is a subset of $SCSG_{\mathbf{G}}$ (eq. 33). Theorem 2 for this case is represented as follows.

$$\widetilde{\mathrm{M}}_{\mathbf{H}_j}\mathrm{L}_{\mathbf{H}_j\uparrow\mathbf{H}} = \widetilde{\widetilde{\mathrm{M}}}_{\mathbf{H}\downarrow\mathbf{H}_j}. \tag{35}$$

Since **H** is a subgroup of **G**, we can partly characterize the group **G** by a set of dominant markaracter tables ($\widetilde{\mathrm{M}}_{\mathbf{H}_j}$) and a set of magnifications ($\mathrm{L}_{\mathbf{H}_j\uparrow\mathbf{G}}$), where $\mathbf{H}_j$ runs over the SCSG of eq. 34. These sets are subset of the corresponding ones for full characterization described above. Thus, we have

$$\widetilde{\mathrm{M}}_{\mathbf{H}_j}\mathrm{L}_{\mathbf{H}_j\uparrow\mathbf{G}} = \widetilde{\widetilde{\mathrm{M}}}_{\mathbf{G}\downarrow\mathbf{H}_j}. \tag{36}$$

Equations 35 and 36 give a theorem.

Theorem 3.

$$\widetilde{\widetilde{\mathrm{M}}}_{\mathbf{G}\downarrow\mathbf{H}_j} = \widetilde{\widetilde{\mathrm{M}}}_{\mathbf{H}\downarrow\mathbf{H}_j}\mathrm{L}^{-1}_{\mathbf{H}_j\uparrow\mathbf{H}}\mathrm{L}_{\mathbf{H}_j\uparrow\mathbf{G}} \tag{37}$$

$$= \widetilde{\widetilde{\mathrm{M}}}_{\mathbf{H}\downarrow\mathbf{H}_j}\mathrm{L}^{(j)}_{\mathbf{H}\uparrow\mathbf{G}} \tag{38}$$

where the maginfication $\mathrm{L}^{(j)}_{\mathbf{H}\uparrow\mathbf{G}}$ *is substituted for* $\mathrm{L}^{-1}_{\mathbf{H}_j\uparrow\mathbf{H}}\mathrm{L}_{\mathbf{H}_j\uparrow\mathbf{G}}$.

The resulting maginifcations from $\mathbf{H}$ to $\mathbf{G}$(*i.e.* $L^{(j)}_{\mathbf{H}\uparrow\mathbf{G}}$ for $j = 1, 2, \ldots, u$) are diagonal matrices. It follows that

$$L_{\mathbf{H}_j\uparrow\mathbf{G}} = L_{\mathbf{H}_j\uparrow\mathbf{H}} L^{(j)}_{\mathbf{H}\uparrow\mathbf{G}}. \tag{39}$$

Since the concrete forms of $L_{\mathbf{H}_j\uparrow\mathbf{H}}$ and $L_{\mathbf{H}_j\uparrow\mathbf{G}}$ are obtained by using eq. 30, we have

$$L^{(j)}_{\mathbf{H}\uparrow\mathbf{G}} = \begin{pmatrix} \frac{|\mathbf{N_G}(\mathbf{G}^{(j)}_1)|}{|\mathbf{N_H}(\mathbf{G}^{(j)}_1)|} & & & 0 \\ & \frac{|\mathbf{N_G}(\mathbf{G}^{(j)}_2)|}{|\mathbf{N_H}(\mathbf{G}^{(j)}_2)|} & & \\ & & \ddots & \\ 0 & & & \frac{|\mathbf{N_G}(\mathbf{G}^{(j)}_r)|}{|\mathbf{N_H}(\mathbf{G}^{(j)}_r)|} \end{pmatrix}, \tag{40}$$

where the cyclic group $\mathbf{G}_j$ is a subgroup of $\mathbf{H}$.

Example 2. For example, markaracter tables for cyclic subgroups $\mathbf{C}_s$ and $\mathbf{C}_3$ are magnified into those for $\mathbf{T}_d$ *via* those for $\mathbf{C}_{3v}$ as follows, where matrices on arrows represent maginifcations.

$$\mathbf{C}_s: \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}} \overset{\mathbf{C}_{3v}}{\begin{pmatrix} 6 & 0 \\ 3 & 1 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}} \overset{\mathbf{T}_d}{\begin{pmatrix} 24 & 0 \\ 12 & 2 \end{pmatrix}} \tag{41a}$$

$$\mathbf{C}_3: \quad \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} \overset{\mathbf{C}_{3v}}{\begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}} \overset{\mathbf{T}_d}{\begin{pmatrix} 24 & 0 \\ 8 & 2 \end{pmatrix}} \tag{41a}$$

□

Example 3. Let us next consider markaracter tables for cyclic subgroups $\mathbf{C}_2$, $\mathbf{C}_s$ and $\mathbf{S}_4$ are magnified into those for $\mathbf{T}_d$ *via* those for $\mathbf{D}_{2d}$, where two non-conjugate groups are fused into the subgroup $\mathbf{C}_2$ of $\mathbf{T}_d$.

$$\mathbf{C}_2: \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}} \overset{\mathbf{D}_{2d}}{\begin{pmatrix} 8 & 0 \\ 4 & 4 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}} \overset{\mathbf{T}_d}{\begin{pmatrix} 24 & 0 \\ 12 & 4 \end{pmatrix}} \tag{42a}$$

$$\mathbf{C}'_2: \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}} \overset{\mathbf{D}_{2d}}{\begin{pmatrix} 8 & 0 \\ 4 & 2 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}} \overset{\mathbf{T}_d}{\begin{pmatrix} 24 & 0 \\ 12 & 4 \end{pmatrix}} \tag{42b}$$

$$\mathbf{C}_s: \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}} \overset{\mathbf{D}_{2d}}{\begin{pmatrix} 8 & 0 \\ 4 & 2 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}} \overset{\mathbf{T}_d}{\begin{pmatrix} 24 & 0 \\ 12 & 2 \end{pmatrix}} \tag{42c}$$

$$\mathbf{S}_4: \quad \begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \overset{\mathbf{D}_{2d}}{\begin{pmatrix} 8 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 2 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\begin{matrix} \mathbf{T}_d \\ \begin{pmatrix} 24 & 0 & 0 \\ 12 & 4 & 0 \\ 6 & 2 & 2 \end{pmatrix} \end{matrix} \quad (42d)$$

It should be noted that the total magnification of $\mathbf{C}_2$ into $\mathbf{T}_d$ is equal to that of $\mathbf{C}_2'$ into $\mathbf{T}_d$. □

3. Subduction and combinatorial enumeration

3.1. Dominant subduction tables and dominant USCI tables

The subduction of coset representations has been reported in previous papers [6, 7]. Dominant representations (coset representaitons for cyclic subgroups) can be subduced in a similar way, where the subduction is concerned with cyclic subgroups. Let $\mathbf{G}_j$ be a cyclic subgroup of $\mathbf{G}$, where the markaracter table is represented by $\widetilde{\widetilde{\mathbf{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$. Consider a modified mark table of $\mathbf{G}$ in which the columns corresponding to $\mathbf{G}_j$ are gathered into the upperleft part by means of consecutive concurrent interchanges. Then, we select the columns corresponding to $\mathbf{G}_j$ from the modified mark table to give an $t \times r$ matrix:

$$\widehat{\mathbf{M}}_{\mathbf{G}\downarrow\mathbf{G}_j} = (\widehat{m}_{i\ell}^{(j)}) =
\begin{array}{c} \\ \mathbf{G}(/\mathbf{G}_1)\downarrow\mathbf{G}_j \\ \mathbf{G}(/\mathbf{G}_2)\downarrow\mathbf{G}_j \\ \vdots \\ \mathbf{G}(/\mathbf{G}_k)\downarrow\mathbf{G}_j \\ \vdots \\ \mathbf{G}(/\mathbf{G}_r)\downarrow\mathbf{G}_j \\ \vdots \\ \mathbf{G}(/\mathbf{G}_i)\downarrow\mathbf{G}_j \\ \vdots \\ \mathbf{G}(/\mathbf{G}_s)\downarrow\mathbf{G}_j \\ \vdots \\ \mathbf{G}(/\mathbf{G}_t)\downarrow\mathbf{G}_j \end{array}
\begin{array}{c} \begin{array}{cccccc} \downarrow\mathbf{G}_1^{(j)} & \downarrow\mathbf{G}_2^{(j)} & \cdots & \downarrow\mathbf{G}_k^{(j)} & \cdots & \downarrow\mathbf{G}_r^{(j)} \end{array} \\
\begin{pmatrix}
\widehat{m}_{11}^{(j)} & & & & & \\
\widehat{m}_{21}^{(j)} & \widehat{m}_{22}^{(j)} & & & & \\
\vdots & \vdots & \ddots & & & \\
\widehat{m}_{k1}^{(j)} & \widehat{m}_{k2}^{(j)} & \cdots & \widehat{m}_{kk}^{(j)} & & \\
\vdots & \vdots & & \vdots & \ddots & \\
\widehat{m}_{r1}^{(j)} & \widehat{m}_{r2}^{(j)} & \cdots & \widehat{m}_{rk}^{(j)} & \cdots & \widehat{m}_{rr}^{(j)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
\widehat{m}_{i1}^{(j)} & \widehat{m}_{i2}^{(j)} & \cdots & \widehat{m}_{ik}^{(j)} & \cdots & \widehat{m}_{ir}^{(j)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
\widehat{m}_{s1}^{(j)} & \widehat{m}_{s2}^{(j)} & \cdots & \widehat{m}_{sk}^{(j)} & \cdots & \widehat{m}_{sr}^{(j)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
\widehat{m}_{t1}^{(j)} & \widehat{m}_{t2}^{(j)} & \cdots & \widehat{m}_{tk}^{(j)} & \cdots & \widehat{m}_{tr}^{(j)}
\end{pmatrix} \end{array}, \quad (43)$$

where zero values are omitted and $\mathbf{G}_k$ is equal to $\mathbf{G}_k^{(j)}$ for $k = 1, 2, \ldots, r$. It is easy to show that the upper $r \times r$ part of the matrix (eq. 43) is identical with the SDMT ($\widetilde{\widetilde{\mathbf{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$). The rows between $r+1$ and s correspond to cyclic subgroups ($\mathbf{G}_{r+1}$ to $\mathbf{G}_s$) and the rows between $s+1$ and t are assigned to non-cyclic subgroups ($\mathbf{G}_{s+1}$ to $\mathbf{G}_t$).

Since Lemma 9.1 of Ref. [7] holds for this case, we have

$$\beta_k^{(ij)} = \sum_{\ell=1}^{r} \widehat{m}_{i\ell}^{(j)} \overline{\lambda}_{\ell k}^{(j)}, \quad (44)$$

where $k = 1, 2, \ldots, r$, $i = 1, 2, \ldots, t$ (tentatively fixed) and $j = 1, 2, \ldots, r$ (tentatively fixed). The term $\overline{\lambda}_{\ell k}^{(j)}$ represents the element of the inverse ($\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}$) of the markaracter table $\widetilde{\mathbf{M}}_{\mathbf{G}_j} = (\lambda_{k\ell}^{(j)})$. Note that r depends on $\mathbf{G}_j$ though not expressed. When we consider the following row vectors,

$$\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j = (\widehat{m}_{i1}^{(j)}, \widehat{m}_{i2}^{(j)}, \cdots, \widehat{m}_{ir}^{(j)}) \tag{45}$$

$$\widetilde{\mathbf{G}}(/\mathbf{G}_i) \downarrow \mathbf{G}_j = (\beta_1^{(ij)}, \beta_2^{(ij)}, \cdots, \beta_r^{(ij)}), \tag{46}$$

eq. 44 is transformed into

$$[\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j]\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} = \widetilde{\mathbf{G}}(/\mathbf{G}_i) \downarrow \mathbf{G}_j. \tag{47}$$

Theorem 9.1 of Ref. [7] holds for this case,

$$\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j = \sum_{k=1}^{r} \beta_k^{(ij)} \mathbf{G}_j(/\mathbf{G}_k^{(j)}), \tag{48}$$

where $i = 1, 2, \ldots, t$ (tentatively fixed) and $j = 1, 2, \ldots, r$ (tentatively fixed).

In order to calculate the subduction multiplicities ($\beta_k^{(ij)}$), we here treat the calculation in three steps:

1. The generation of a dominant subduction table and of a dominant USCI table, where $\mathbf{G}_i$ of $\mathbf{G}(/\mathbf{G}_i)$ is cyclic (Case 1).
2. The subduction of $\mathbf{G}(/\mathbf{G}_i)$ in which $\mathbf{G}_i$ is non-cyclic (Case 2).
3. An alternative evaluation of Case 2 from the data of Case 1.

Let us first consider cyclic subgroups only (Case 1). In other words, we take into consideration the first to the s-th row of the matrix $\widehat{\mathbf{M}}_{\mathbf{G}\downarrow\mathbf{G}_j}$ (eq. 43), where $\mathbf{G}_j$ runs from $j = 1$ to s. The calculation of the subduction multiplicities by eq. 44 is conveniently conducted by fixing $\mathbf{G}_j$. Thus, we move k of $\beta_k^{(ij)}$ from 1 to r and the i from 1 to s so that we have an $s \times r$ matrix, *i.e.*, $\widehat{\widetilde{\mathbf{B}}}_j = (\beta_k^{(ij)})$, which is called a *subduction-multiplicity matrix*. In terms of this definition, we obtain the matrix $\widehat{\widetilde{\mathbf{B}}}_j$ by solving eq. 48, which is explicitly represented as follows.

$$\widehat{\widetilde{\mathbf{B}}}_j = \widehat{\widehat{\mathbf{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}, \tag{49}$$

where the matrix $\widehat{\widehat{\mathbf{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$ is the upper $s \times r$ part of eq. 43. Note that the i-th row of $\widehat{\widetilde{\mathbf{B}}}_j$ gives the coefficients for each $\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j$. When the j of the matrix $\widehat{\widetilde{\mathbf{B}}}_j$ runs from 1 to s, we have a dominant subduction table of the group $\mathbf{G}$.

In accord with eq. 48, the unit subduced cycle index (USCI) for $\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j$ is defined as

$$Z(\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j; s_{d_{jk}}) = \prod_{k=1}^{r} s_{d_{jk}}^{\beta_k^{(ij)}}, \tag{50}$$

where each subscript is calculated by

$$d_{jk} = \frac{|\mathbf{G}_j|}{|\mathbf{G}_k^{(j)}|}. \tag{51}$$

Example 4. Let us evaluate $\mathbf{T}_d(/\mathbf{G}_i) \downarrow \mathbf{S}_4$, where $\mathbf{G}_i$ is selected to be $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_s$, $\mathbf{C}_3$, and $\mathbf{S}_4$. We make a subduced markaracter table by selecting necessary columns ($\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{S}_4$) from the markaracter table of the group $\mathbf{T}_d$. The resulting matrix is multiplied by the inverse ($\widetilde{\mathbf{M}}_{\mathbf{S}_4}^{-1}$) to give a subduction-multiplicity matrix, *i.e.*,

$$\begin{array}{c} \\ \mathbf{T}_d(/\mathbf{C}_1) \\ \mathbf{T}_d(/\mathbf{C}_2) \\ \mathbf{T}_d(/\mathbf{S}_4) \\ \mathbf{T}_d(/\mathbf{C}_s) \\ \mathbf{T}_d(/\mathbf{C}_3) \end{array}
\begin{array}{c} \begin{array}{ccc} \downarrow\mathbf{C}_1 & \downarrow\mathbf{C}_2 & \downarrow\mathbf{S}_4 \end{array} \\ \begin{pmatrix} 24 & 0 & 0 \\ 12 & 4 & 0 \\ 6 & 2 & 2 \\ 12 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix} \end{array}
\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$= \begin{array}{c} \\ \mathbf{T}_d(/\mathbf{C}_1) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_2) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{S}_4) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_s) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_3) \downarrow \mathbf{S}_4 \end{array}
\begin{array}{c} \begin{array}{ccc} \downarrow\mathbf{C}_1 & \downarrow\mathbf{C}_2 & \downarrow\mathbf{S}_4 \end{array} \\ \begin{pmatrix} 6 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \end{array}
\begin{array}{l} \\ s_4^6 \\ s_4^2 s_2^2 \\ s_1^2 s_4 \\ s_4^3 \\ s_4^3 \end{array} \tag{52}$$

The resulting matrix contains the multiplicities for the respective subductions. For example, the second row of the matrix means

$$\mathbf{T}_d(/\mathbf{C}_2) \downarrow \mathbf{S}_4 = 2\mathbf{S}_4(/\mathbf{C}_1) + 2\mathbf{S}_4(/\mathbf{C}_2), \tag{53}$$

which generates a USCI, $s_4^2 s_2^2$, since $|\mathbf{S}_4|/|\mathbf{C}_1| = 4$ and $|\mathbf{S}_4|/|\mathbf{C}_2| = 2$.

The procedure of this example is repeated for each dominant representation (DR) to give a dominant subduction table (Table 1) and a dominant USCI table (Table 2). □

Comparison between eqs. 24 and 52 gives the following relationship:

$$\begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 0 & 0 \\ 12 & 4 & 0 \\ 6 & 2 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \tag{54}$$

The first matrix in the left-hand side is inverse to the third one. This relationship reveals the relationship between the magnification and the subduction multiplicities. The relationship is easily extended to a general case. Let $\widetilde{\mathbf{B}}_j = (\beta_k^{(ij)})$, in which k runs over the columns from 1 to r and i runs over the rows from 1 to r (j is tentatively fixed). Then, we have

$$\widetilde{\mathbf{M}}_{\mathbf{G}_j} \mathbf{L}_{\mathbf{G}_j \uparrow \mathbf{G}} \widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} = \widetilde{\mathbf{B}}_j. \tag{55}$$

Table 1. Dominant subduction table of $\mathbf{T}_d$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_s$	$\downarrow \mathbf{C}_3$	$\downarrow \mathbf{S}_4$
$\mathbf{T}_d(/\mathbf{C}_1)$	$24\mathbf{C}_1(/\mathbf{C}_1)$	$12\mathbf{C}_2(/\mathbf{C}_1)$	$12\mathbf{C}_s(/\mathbf{C}_1)$	$8\mathbf{C}_3(/\mathbf{C}_1)$	$6\mathbf{S}_4(/\mathbf{C}_1)$
$\mathbf{T}_d(/\mathbf{C}_2)$	$12\mathbf{C}_1(/\mathbf{C}_1)$	$4\mathbf{C}_2(/\mathbf{C}_1)$ $+4\mathbf{C}_2(/\mathbf{C}_2)$	$6\mathbf{C}_s(/\mathbf{C}_1)$	$4\mathbf{C}_3(/\mathbf{C}_1)$	$2\mathbf{S}_4(/\mathbf{C}_1)$ $+2\mathbf{S}_4(/\mathbf{C}_2)$
$\mathbf{T}_d(/\mathbf{C}_s)$	$12\mathbf{C}_1(/\mathbf{C}_1)$	$6\mathbf{C}_2(/\mathbf{C}_1)$	$5\mathbf{C}_s(/\mathbf{C}_1)$ $+2\mathbf{C}_s(/\mathbf{C}_s)$	$4\mathbf{C}_3(/\mathbf{C}_1)$	$3\mathbf{S}_4(/\mathbf{C}_1)$
$\mathbf{T}_d(/\mathbf{C}_3)$	$8\mathbf{C}_1(/\mathbf{C}_1)$	$4\mathbf{C}_2(/\mathbf{C}_1)$	$4\mathbf{C}_s(/\mathbf{C}_1)$	$2\mathbf{C}_3(/\mathbf{C}_1)$ $+2\mathbf{C}_3(/\mathbf{C}_3)$	$2\mathbf{S}_4(/\mathbf{C}_1)$
$\mathbf{T}_d(/\mathbf{S}_4)$	$6\mathbf{C}_1(/\mathbf{C}_1)$	$2\mathbf{C}_2(/\mathbf{C}_1)$ $+2\mathbf{C}_2(/\mathbf{C}_2)$	$3\mathbf{C}_s(/\mathbf{C}_1)$	$2\mathbf{C}_3(/\mathbf{C}_1)$	$\mathbf{S}_4(/\mathbf{C}_1)$ $+2\mathbf{S}_4(/\mathbf{S}_4)$

Table 2. Dominant USCI table of $\mathbf{T}_d$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_s$	$\downarrow \mathbf{C}_3$	$\downarrow \mathbf{S}_4$
$\mathbf{T}_d(/\mathbf{C}_1)$	s_1^{24}	s_2^{12}	s_2^{12}	s_3^8	s_4^6
$\mathbf{T}_d(/\mathbf{C}_2)$	s_1^{12}	$s_1^4 s_2^4$	s_6^2	s_3^4	$s_2^2 s_4^2$
$\mathbf{T}_d(/\mathbf{C}_s)$	s_1^{12}	s_2^6	$s_1^2 s_2^5$	s_3^4	s_4^3
$\mathbf{T}_d(/\mathbf{C}_3)$	s_1^8	s_2^4	s_2^4	$s_1^2 s_3^2$	s_4^2
$\mathbf{T}_d(/\mathbf{S}_4)$	s_1^6	$s_1^2 s_2^2$	s_2^3	s_3^2	$s_1^2 s_4$

3.2. Subduction of non-dominant representations

A coset representation which is not a DR is called a non-dominant representation. For discussing the subduction of such an non-dominant representation, we consider an $t \times r$ matrix, $\widehat{\mathrm{B}}_j = (\beta_k^{(ij)})$, in which k runs over the columns from 1 to r and i runs over the rows from 1 to t (j is tentatively fixed). This matrix contains the $\widetilde{\widehat{\mathrm{B}}}_j$ as the upper s rows.

Then, eq. 44 is transformed into

$$\widehat{\mathrm{B}}_j = \widehat{\mathrm{M}}_{\mathbf{G}\downarrow\mathbf{G}_j} \widetilde{\mathrm{M}}_{\mathbf{G}_j}^{-1} \tag{56}$$

This treatment turns out to take account of Case 2, if we focus our attention on the $(s+1)$-the to the t-th rows of $\widehat{\mathrm{M}}_{\mathbf{G}\downarrow\mathbf{G}_j}$ (eq. 43) and on the corresponding part of B. For illustrating the non-cyclic part of $\widehat{\mathrm{B}}_j = (\beta_k^{(ij)})$, we examine $\mathbf{T}_d(/\mathbf{G}_i) \downarrow \mathbf{S}_4$, where $\mathbf{G}_i$ is a non-cyclic subgroup of $\mathbf{T}_d$.

Example 5. Let us evaluate $\mathbf{T}_d(/\mathbf{G}_i) \downarrow \mathbf{S}_4$, where $\mathbf{G}_i$ is selected to be $\mathbf{D}_2$, $\mathbf{C}_{2v}$, $\mathbf{C}_{3v}$, $\mathbf{D}_{2d}$, $\mathbf{T}$, and $\mathbf{T}_d$. We make a subduced markaracter table by selecting necessary columns ($\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{S}_4$) from the mark table of the group $\mathbf{T}_d$. The resulting matrix is multiplied by the inverse ($\widetilde{\mathrm{M}}_{\mathbf{S}_4}^{-1}$) to give

$$\begin{array}{l} \\ \mathbf{T}_d(/\mathbf{D}_4) \\ \mathbf{T}_d(/\mathbf{C}_{2v}) \\ \mathbf{T}_d(/\mathbf{C}_{3v}) \\ \mathbf{T}_d(/\mathbf{D}_{2d}) \\ \mathbf{T}_d(/\mathbf{T}) \\ \mathbf{T}_d(/\mathbf{T}_d) \end{array} \begin{array}{c} \begin{array}{ccc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{S}_4 \end{array} \\ \begin{pmatrix} 6 & 6 & 0 \\ 6 & 2 & 0 \\ 4 & 0 & 0 \\ 3 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$= \begin{array}{l} \\ \mathbf{T}_d(/\mathbf{D}_2) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_{2v}) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_{3v}) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{D}_{2d}) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{T}) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{T}_d) \downarrow \mathbf{S}_4 \end{array} \begin{array}{c} \begin{array}{ccc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{S}_4 \end{array} \\ \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ s_2^3 \\ s_2 s_4 \\ s_4 \\ s_1 s_2 \\ s_2 \\ s_1 \end{array} \quad (57)$$

The resulting matrix gives the multiplicities for the respective subductions. For example, the second row of the matrix means

$$\mathbf{T}_d(/\mathbf{C}_{2v}) \downarrow \mathbf{S}_4 = \mathbf{S}_4(/\mathbf{C}_1) + \mathbf{S}_4(/\mathbf{C}_2), \quad (58)$$

which generates a USCI, $s_2 s_4$, since $|\mathbf{S}_4|/|\mathbf{C}_1| = 4$ and $|\mathbf{S}_4|/|\mathbf{C}_2| = 2$. The variables in the rightmost column of eq. 57 are USCIs for the subduction $\mathbf{T}_d(/\mathbf{G}_i) \downarrow \mathbf{S}_4$.

The procedure of this example is repeated for each non-dominant representation to give a subduction table (Table 3) and a USCI table (Table 4) for non-cyclic subgroups. □

The last step is to derive Case 2 from the data of Case 1. Let us consider a coset representation $\mathbf{G}(/\mathbf{G}_i)$ in which $\mathbf{G}_i$ is a non-cyclic subgroup of $\mathbf{G}$. Suppose that $\mathbf{G}(/\mathbf{G}_i)$ is associated with the following markaracter and multiplicity vector:

$$\mathbf{G}(/\mathbf{G}_i) = (x_1, x_2, \ldots, x_s) \quad (59)$$

$$\widetilde{\mathbf{A}} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_s). \quad (60)$$

In the light of the discussions described in the preceding article, the latter is obtained from the former by means of the equation:

$$\mathbf{G}(/\mathbf{G}_i)\widetilde{\mathbf{M}}_{\mathbf{G}}^{-1} = \widetilde{\mathbf{A}}. \quad (61)$$

Equations 49 and 61 give

$$\begin{aligned} \widetilde{\mathbf{A}}\widehat{\widetilde{\mathbf{B}}}_j &= \mathbf{G}(/\mathbf{G}_i)\widetilde{\mathbf{M}}_{\mathbf{G}}^{-1}\widehat{\widetilde{\mathbf{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} \\ &= \mathbf{G}(/\mathbf{G}_i) \begin{pmatrix} 1 & & & \\ & 1 & & \mathbf{0} \\ \mathbf{0} & & \ddots & \\ & & & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1}, \end{aligned}$$

Table 3. Subduction of non-dominant CRs for $\mathbf{T}_d$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_s$	$\downarrow \mathbf{C}_3$	$\downarrow \mathbf{S}_4$
$\mathbf{T}_d(/\mathbf{D}_2)$	$6\mathbf{C}_1(/\mathbf{C}_1)$	$6\mathbf{C}_2(/\mathbf{C}_2)$	$3\mathbf{C}_s(/\mathbf{C}_1)$	$2\mathbf{C}_3(/\mathbf{C}_1)$	$3\mathbf{S}_4(/\mathbf{C}_2)$
$\mathbf{T}_d(/\mathbf{C}_{2v})$	$6\mathbf{C}_1(/\mathbf{C}_1)$	$2\mathbf{C}_2(/\mathbf{C}_1)$ $+2\mathbf{C}_2(/\mathbf{C}_2)$	$2\mathbf{C}_s(/\mathbf{C}_1)$ $+2\mathbf{C}_s(/\mathbf{C}_s)$	$2\mathbf{C}_3(/\mathbf{C}_1)$	$\mathbf{S}_4(/\mathbf{C}_1)$ $+\mathbf{S}_4(/\mathbf{C}_2)$
$\mathbf{T}_d(/\mathbf{C}_{3v})$	$4\mathbf{C}_1(/\mathbf{C}_1)$	$2\mathbf{C}_2(/\mathbf{C}_1)$	$\mathbf{C}_s(/\mathbf{C}_1)$ $+2\mathbf{C}_s(/\mathbf{C}_s)$	$\mathbf{C}_3(/\mathbf{C}_1)$ $+\mathbf{C}_3(/\mathbf{C}_3)$	$\mathbf{S}_4(/\mathbf{C}_1)$
$\mathbf{T}_d(/\mathbf{D}_{2d})$	$3\mathbf{C}_1(/\mathbf{C}_1)$	$3\mathbf{C}_2(/\mathbf{C}_2)$	$\mathbf{C}_s(/\mathbf{C}_1)$ $+2\mathbf{C}_s(/\mathbf{C}_s)$	$\mathbf{C}_3(/\mathbf{C}_1)$	$\mathbf{S}_4(/\mathbf{C}_2)$ $+\mathbf{S}_4(/\mathbf{S}_4)$
$\mathbf{T}_d(/\mathbf{T})$	$2\mathbf{C}_1(/\mathbf{C}_1)$	$2\mathbf{C}_2(/\mathbf{C}_2)$	$\mathbf{C}_s(/\mathbf{C}_1)$	$2\mathbf{C}_3(/\mathbf{C}_3)$	$\mathbf{S}_4(/\mathbf{C}_2)$
$\mathbf{T}_d(/\mathbf{T}_d)$	$\mathbf{C}_1(/\mathbf{C}_1)$	$\mathbf{C}_2(/\mathbf{C}_2)$	$\mathbf{C}_s(/\mathbf{C}_s)$	$\mathbf{C}_3(/\mathbf{C}_3)$	$\mathbf{S}_4(/\mathbf{S}_4)$

Table 4. USCIs for non-dominant representations of $\mathbf{T}_d$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_s$	$\downarrow \mathbf{C}_3$	$\downarrow \mathbf{S}_4$
$\mathbf{T}_d(/\mathbf{D}_2)$	s_1^6	s_1^6	s_2^4	s_3^2	s_2^3
$\mathbf{T}_d(/\mathbf{C}_{2v})$	s_1^6	$s_1^2 s_2^2$	$s_1^2 s_2^2$	s_3^2	$s_2 s_4$
$\mathbf{T}_d(/\mathbf{C}_{3v})$	s_1^4	s_2^2	$s_1^2 s_2$	$s_1 s_3$	s_4
$\mathbf{T}_d(/\mathbf{D}_{2d})$	s_1^3	s_1^3	$s_1 s_2$	s_3	$s_1 s_2$
$\mathbf{T}_d(/\mathbf{T})$	s_1^2	s_1^2	s_2	s_1^2	s_2
$\mathbf{T}_d(/\mathbf{T}_d)$	s_1	s_1	s_1	s_1	s_1

where the middle matrix of a concrete form is obtained by considering the fact that $\widetilde{\mathbf{M}}_{\mathbf{G}}^{-1}$ is inverse to the markaracter table $\widetilde{\mathbf{M}}_{\mathbf{G}}$ which involves $\widehat{\widetilde{\mathbf{M}}}_{\mathbf{G}\downarrow\mathbf{G}_j}$ as the left r columns. Since the rows from $r+1$ to s in the matrix $\widehat{\widetilde{\mathbf{B}}}_j$ vanish in this process, we can use the matrix $\widetilde{\mathbf{B}}_j$ that contains the first to r-th rows of the $\widehat{\widetilde{\mathbf{B}}}_j$. The vector $\widetilde{\mathbf{A}}$ is transformed into the restricted one ($\widetilde{\mathbf{A}}'$) concerning $\mathbf{G}_j$. It follows that

$$\begin{aligned}\widetilde{\mathbf{A}}'\widetilde{\mathbf{B}}_j &= [\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j]\widetilde{\mathbf{M}}_{\mathbf{G}_j}^{-1} \\ &= \widetilde{\mathbf{G}}(/\mathbf{G}_i) \downarrow \mathbf{G}_j. \end{aligned} \tag{62}$$

The last transformation stems from eq. 47. Equation 62 can be transformed into an equivalent equation by using $\widetilde{\mathbf{A}}$ and $\widehat{\widetilde{\mathbf{B}}}_j$, where zero values are added to $\widetilde{\mathbf{A}}'$ for regenerating $\widetilde{\mathbf{A}}$. When we explicitly express the row vectors, we have

$$\widetilde{\mathbf{A}}\widehat{\widetilde{\mathbf{B}}}_j = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_s)\widehat{\widetilde{\mathbf{B}}}_j = (\beta_1^{(ij)}, \beta_2^{(ij)}, \cdots, \beta_r^{(ij)}) = \widetilde{\mathbf{G}}(/\mathbf{G}_i) \downarrow \mathbf{G}_j,$$

where $i = r+1, r+2, \ldots s$ and j is tentatively fixed. It should be noted that $\mathbf{G}_i$ is a non-cyclic subgroup of $\mathbf{G}$ while $\widetilde{\mathbf{A}}$ and $\widehat{\widetilde{\mathbf{B}}}_j$ are concerned with the cyclic subgroup $\mathbf{G}_j$. Moreover, $\widehat{\widetilde{\mathbf{B}}}_j$ ($j = 1, 2, \ldots, s$) creats the dominant subduction table and the dominant USCI table. This means that subduction for non-dominant representations can be evaluated from the data of the dominant subduction table and of the dominant USCI table. We summarizes the result as a theorem.

Theorem 4. *Suppose that the multiplicity vector* $\widetilde{\mathbf{A}}$ *is obtained from* $\mathbf{G}(/\mathbf{G}_i)$ *by eq. 60 and that the subduction-multiplicity matrix* $\widehat{\widetilde{\mathbf{B}}}_j$ *is obtained by eq. 49. Then* $\widetilde{\mathbf{G}}(/\mathbf{G}_i) \downarrow \mathbf{G}_j$ *is obtained by calculating*

$$\widetilde{\mathbf{A}}\widehat{\widetilde{\mathbf{B}}}_j = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_s)\widehat{\widetilde{\mathbf{B}}}_j = (\beta_1^{(ij)}, \beta_2^{(ij)}, \cdots, \beta_r^{(ij)}) = \widetilde{\mathbf{G}}(/\mathbf{G}_i) \downarrow \mathbf{G}_j, \qquad (63)$$

for $j = 1, 2, \ldots, s$.

Example 6. From the markaracater $\mathbf{T}_d(/\mathbf{C}_{3v}) = (4, 0, 2, 1, 0)$, we have a multiplicity vector,

$$\widetilde{\mathbf{T}}_d(/\mathbf{C}_{3v}) = (4, 0, 2, 1, 0)\widetilde{\mathbf{M}}_{\mathbf{T}_d}^{-1} = (-\frac{1}{2}, 0, 1, \frac{1}{2}, 0). \qquad (64)$$

Thereby, $\mathbf{T}_d(/\mathbf{C}_{3v}) \downarrow \mathbf{S}_4$ is calculated as follows from the data in the $\mathbf{S}_4$-column of Table 1,

$$\begin{aligned}\mathbf{T}_d(/\mathbf{C}_{3v}) \downarrow \mathbf{S}_4 &= -\frac{1}{2} \times (6\mathbf{S}_4(/\mathbf{C}_1)) + 3\mathbf{S}_4(/\mathbf{C}_1) + \frac{1}{2} \times (2\mathbf{C}_1(/\mathbf{C}_1)) \\ &= \mathbf{S}_4(/\mathbf{C}_1) \qquad (65)\end{aligned}$$

This result is identical with the corresponding element of Table 3.

On the other hand, the multiplicity vector gives the USCI for $\mathbf{T}_d(/\mathbf{C}_{3v}) \downarrow \mathbf{S}_4$ from the data in the $\mathbf{S}_4$-column of Table 2,

$$s_4^{6\times(-\frac{1}{2})} \times s_4^{3\times 1} \times s_4^{2\times\frac{1}{2}} = s_4 \qquad (66)$$

This result is identical with the corresponding element of Table 4.

The procedures are based on eq. 63. This fact can be verified: eq. 63 for the present example is calculated to be

$$\widetilde{\mathbf{T}}_d(/\mathbf{C}_{3v}) \downarrow \mathbf{S}_4 =$$

$$(-\frac{1}{2}, 0, 1, \frac{1}{2}, 0)\begin{matrix} \mathbf{T}_d(/\mathbf{C}_1) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_2) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_s) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{C}_3) \downarrow \mathbf{S}_4 \\ \mathbf{T}_d(/\mathbf{S}_4) \downarrow \mathbf{S}_4 \end{matrix} \overset{\begin{matrix}\downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{S}_4\end{matrix}}{\begin{pmatrix} 6 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}} = (1, 0, 0) \qquad (67)$$

□

The treatment concerning eqs. 59 and 60 can be extended into a general case, in which $\mathbf{G}(/\mathbf{G}_i)$ is replaced by a markaracter $\mathbf{P} = (x_1, x_2, \ldots, x_s)$. This extention shall be discussed for the new formulation of combinatorial enumeration in the next section.

3.3. Combinatorial enumeration

Consider a skeleton with n positions, which belongs to point group $\mathbf{G}$. The group $\mathbf{G}$ has a non-redundant set of dominant subgroups,

$$SCSG_{\mathbf{G}} = \{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_s\}. \tag{68}$$

The positions are characterized by a permutation representation $\mathbf{P}$ ($n = |\mathbf{P}|$) with a markaracter, $\mathbf{P} = (\delta_1, \delta_2, \ldots, \delta_s)$. This vector generates a multiplicity vector, $\widetilde{\mathbf{A}} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_s)$, by solving a set of linear equations,

$$\widetilde{\mathbf{A}} = \mathbf{P}\widetilde{\mathbf{M}}_{\mathbf{G}}^{-1}. \tag{69}$$

The multiplicity vector corresponds to

$$\mathbf{P} = \sum_{i=1}^{s} \widetilde{\alpha}_i \mathbf{G}(/\mathbf{G}_i). \tag{70}$$

Let us consider isomers based on the skeleton with n ligands selected from a ligand set:

$$\mathbf{X} = \{X_1, X_2, \ldots, X_v\}. \tag{71}$$

Then, our problem is to obtain the number (A_θ) of isomers with formula:

$$W_\theta = X_1^{\theta_1} X_n^{\theta_2} \cdots X_v^{\theta_v}, \tag{72}$$

where $[\theta]$ represents a partition:

$$[\theta] : \theta_1 + \theta_2 + \cdots + \theta_v = n. \tag{73}$$

In order to apply the present formulation to combinatorial enumeration, we define a cycle index:

$$\begin{aligned} CI(\mathbf{G}; s_{d_{jk}}) &= \sum_{j=1}^{s} \left(\left(\sum_{i=1}^{s} \overline{m}_{ji} \right) \prod_{i=1}^{s} \left(Z(\mathbf{G}(/\mathbf{G}_i) \downarrow \mathbf{G}_j; s_{d_{jk}}) \right)^{\widetilde{\alpha}_i} \right) \\ &= \sum_{j=1}^{s} \left(\left(\sum_{i=1}^{s} \overline{m}_{ji} \right) \prod_{i=1}^{s} \left(\prod_{k=1}^{r_j} s_{d_{jk}}^{\beta_k^{(ij)}} \right)^{\widetilde{\alpha}_i} \right) \\ &= \sum_{j=1}^{s} \left(\left(\sum_{i=1}^{s} \overline{m}_{ji} \right) \prod_{k=1}^{r_j} s_{d_{jk}}^{\beta_k^{(j)}} \right), \end{aligned} \tag{74}$$

where the power $\beta_k^{(j)}$ is represented by

$$\beta_k^{(j)} = \sum_{i=1}^{s} \widetilde{\alpha}_i \beta_k^{(ij)}. \tag{75}$$

The CI can be proved to be identical with the counterpart described in Definition 16.2 in Ref. [7], because the monomials concerning non-cyclic subgroups vanish in the previous formulation. Hence, Theorem 16.1 in Ref. [7] holds in the present formulation, *i.e.*

Theorem 5. *A generaing function for the total number of isomers is represented by*

$$\sum_{[\theta]} A_\theta W_\theta = CI(\mathbf{G}; s_{d_{jk}}), \tag{76}$$

where the right-hand side is substituted by ligand inventories,

$$s_{d_{jk}} = \sum_{\ell=1}^{v} X_\ell^{d_{jk}}. \tag{77}$$

This theorem is based on the cyclic subgroups of **G** while Theorem 16.1 in Ref. [7] takes account of cyclic and non-cyclic subgroups of **G**. However, these two theorems are equivalent to each other. This fact reveals an important role of cyclic subgroups in combinatorial enumeration. Both Theorem 5 (based on dominant USCI tables) and Theorem 16.1 in Ref. [7] (based on USCI tables) are alternative formulations of the Redfield-Pólya theorem (based on cycle structures of respective elements), as illustrated in the following example.

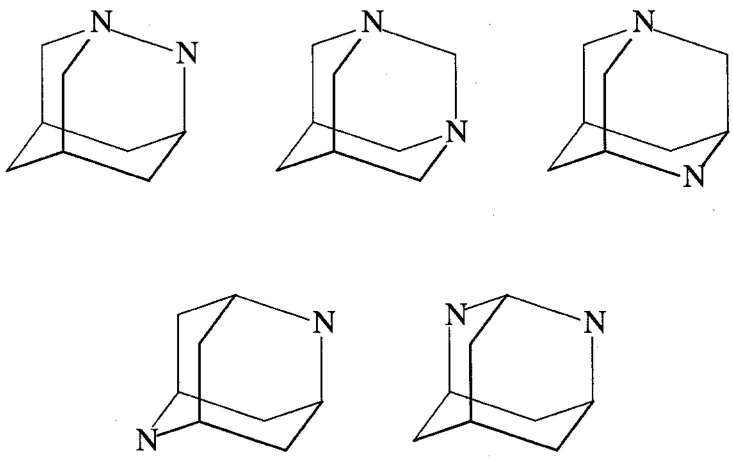

Fig. 1. Five Diaza-adamantanes

Example 7. Let us consider an adamantane skeleton in which we take account of four bridgehead and six bridge positions. These positions are substituted by C or N to produce polyaza-adamantanes. The total 10 positions are characterized by a markaracter $(10, 2, 4, 1, 0)$, which is multiplied by the inverse $(\widetilde{\mathrm{M}}_{\mathbf{T}_d}^{-1})$, *i.e.*,

$$(10, 2, 4, 1, 0)\widetilde{\mathrm{M}}_{\mathbf{T}_d}^{-1} = (-1, \frac{1}{2}, 2, \frac{1}{2}, 0). \tag{78}$$

The orbit index of this multiplicity vector is calculated to be $\widetilde{\Delta} = -1 + \frac{1}{2} + 2 + \frac{1}{2} + 0 = 2$. By using the data of Table 2, eq. 74 for this example is obtained, *i.e.*,

$$
\begin{aligned}
f &= CI(\mathbf{T}_d; s_d) \\
&= \frac{1}{24} s_1^{24\times(-1)+12\times\frac{1}{2}+12\times 2+8\times\frac{1}{2}} + \frac{1}{8} s_1^{4\times\frac{1}{2}} s_2^{12\times(-1)+4\times\frac{1}{2}+6\times 2+4\times\frac{1}{2}} \\
&\quad + \frac{1}{4} s_1^{2\times 2} s_2^{12\times(-1)+6\times\frac{1}{2}+5\times 2+4\times\frac{1}{2}} + \frac{1}{3} s_1^{2\times\frac{1}{2}} s_3^{8\times(-1)+4\times\frac{1}{2}+4\times 2+2\times\frac{1}{2}} \\
&\quad + \frac{1}{4} s_2^{2\times\frac{1}{2}} s_4^{6\times(-1)+2\times\frac{1}{2}+3\times 2+2\times\frac{1}{2}} \\
&= \frac{1}{24} s_1^{10} + \frac{1}{8} s_1^2 s_2^4 + \frac{1}{4} s_1^4 s_2^3 + \frac{1}{3} s_1 s_3^3 + \frac{1}{4} s_2 s_4^2
\end{aligned} \tag{79}
$$

A ligand inventory is obtained to be

$$s_d = C^d + N^d, \tag{80}$$

which is introduced into eq. 79.

$$
\begin{aligned}
f &= \frac{1}{24}(C+N)^{10} + \frac{1}{8}(C+N)^2(C^2+N^2)^4 + \frac{1}{4}(C+N)^4(C^2+N^2)^3 \\
&\quad + \frac{1}{3}(C+N)(C^3+N^3)^3 + \frac{1}{4}(C^2+N^2)(C^4+N^4)^2 \\
&= C^{10} + 2C^9N + 5C^8N^2 + 11C^7N^3 + 17C^6N^{24} + \cdots + N^{10}
\end{aligned} \tag{81}
$$

For illustrating this enumeration, Fig. 1 shows five diaza-adamantans corresponding to the coefficient of the term C^8N^2.

4. Conclusion

The elements of a markaracter table of a cyclic group are evaluated by using the orders of its subgroups. A group **G** of finite order is characterized by the corresponding markaracter table which is constructed with respect to dominant representations (DRs) corresponding to cyclic subgroups (defined as dominant subgroups). The markaracter table is proved to involve an essential set of subdominant markaracter tables which are related to a set of markaracter tables for the dominant subgroups by using a magnification set involving diagonal matrices. The subduction of DRs is obtained by staring the markaracter table. Thereby, we have a dominant subduction table and a dominant USCI (unit-subduced cycle index) table for the group **G**. The dominant USCI table is used to evaluate a cycle index and applied to combinatorial enumeration.

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